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Asymptotically optimal gossiping in radio networks☆

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Abstract

We study the problem of gossiping in a system where n nodes are placed on a line of length L_n independently uniformly distributed. We assume that every node is equipped with a transmitter whose radius of transmission is 1. We further assume that simultaneous transmissions by neighboring nodes results in garbled messages.

We present an algorithm for gossiping and show that it works in asymptotically optimal time. We assume that the system is synchronous and time is slotted and that nodes transmit only during a slot.

1. Introduction

Gossiping is a problem related to information dissemination in communication networks. In gossiping every node in the system has a piece of information that needs to be communicated to everyone else. Communication between a pair of nodes has traditionally been modelled as a *telephone call* during which the two nodes exchange all the information each of them has collected thus far.

Gossiping algorithms have been studied for a variety of system models; see [4] for a comprehensive survey. In the basic system model it is assumed that any node may call any other node. If we represent each *possible* call by an edge in a graph whose n vertices represent the n nodes in the system, then the basic model assumes a *complete* graph. Another assumption is that a node may be engaged in only one call at a time. Given these constraints it is of interest to determine the *shortest sequence of calls* and the *minimum number of time steps* required to complete gossiping (note that several calls may be made simultaneously).

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Several graph-based generalizations of the basic model have been studied. For example, gossiping on hypergraphs [6], grid graphs [3] and trees [11]. Placing restrictions upon the allowable *sequence of calls* yield further generalizations called the NODUP model (no duplication) [13] and the NOHO model (no one hears own) [14, 9].

In this paper we depart significantly from the graph-based models. We assume that nodes are equipped with radio transmitters. Thus when a node transmits, *all* nodes within range hear the message (barring situations where simultaneous transmissions lead to garbled messages). This model is thus very different from models where pairs of nodes communicate via telephone calls. It is noteworthy that our model is relevant to mobile radio networks where gossiping is used to exchange connectivity information.

In this paper we focus our attention on the one-dimensional case where all the nodes are placed on a line. We assume that the system consists of n nodes arranged uniformly *randomly* on a *line* of length L_n. Furthermore, we assume that every node is equipped with a radio transmitter with a transmission radius 1. A message transmitted by a node is received simultaneously by all the nodes within range. If two transmissions reach a node simultaneously then we assume that neither transmission is correctly received by that node (in other words the transmissions have *collided*).

We have studied the problem of *broadcasting* in an earlier paper [7]. We presented an optimal broadcasting algorithm there and we will be utilizing some of those results in the analysis of our gossiping algorithm.

In this paper we present a *distributed* algorithm to gossip when n nodes are placed *uniformly randomly* on a *line* of length L_n. We show that the algorithm is asymptotically optimal, i.e., for a given $\varepsilon > 0$, for large enough $n(\varepsilon)$ (number of nodes) we construct a gossiping algorithm for which the gossiping time is within $(1 + \varepsilon)$ times the optimal in probability. In an earlier paper [10] we presented a *centralized* algorithm to determine *gossiping schedules* for similar systems.

The algorithm presented here is distributed in the sense that every node only needs to know the location of nodes within transmission range to execute the algorithm. In the appendix we address the issue of how nodes go about acquiring this information.

We assume that the topology of the system remains fixed during the execution of the gossiping algorithm. This assumption is appropriate in systems where the topology changes at a much slower rate as compared with the time needed to gossip. We are currently developing algorithms that work when this assumption is not true, i.e., the topology changes very rapidly.

1.1. Outline

The paper is organized as follows. In the next section we provide formal definitions for our system model and define the metrics of interest. Section 3 presents a summary of our results on *broadcasting* that are used in our gossiping algorithm. Section 4 presents the gossiping algorithm and we show that it is asymptotically optimal in

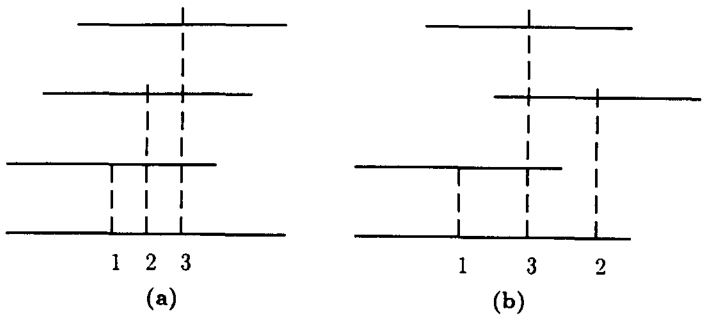


Fig. 1. Node 3 hears noise if 1 and 2 transmit simultaneously.

Section 5. We present our conclusions and suggestions for future work in Section 6. In the appendix we discuss some related implementation issues.

2. The model

Nodes are placed on a line and every node has a transmitter with a transmission radius 1. Nodes i and j are within range of each other if the distance between them is less than 1. Any particular arrangement of the nodes on the line is made up of components each of which is a *connected configuration.* A connected configuration is one where each node is reachable from every other via a series of transmissions. The gossiping algorithm presented in this paper works on a connected configuration.

A natural model for placing nodes with uniform density on a line is a Poisson process. If we restrict the Poisson process to n nodes on a region of length L_n then the distribution of the positions of the nodes are independently identically distributed (i.i.d.) uniform random variables on $[0, L_n]$. We chose this as the model for placement of nodes on the line.

A final aspect of our model deals with the problem of simultaneous transmissions. If a node receives two or more transmissions at the same time then it hears noise, i.e., a collision has occurred. The transmitting nodes however may or may not be aware that a collision has occurred. Consider the following two collision scenerios. The first kind of collision occurs when two or more nodes that are within transmission range of each other (i.e., the distance between them is less than 1) transmit simultaneously, see Fig. 1(a). All the nodes hear noise. The second kind of collision is possible when three nodes 1, 2 and 3 are placed in such a way that node 3 is within range of both 1 and 2. If 1 and 2 transmit simultaneously, node 3 hears noise, however neither 1 nor 2 is aware that 3 heard noise because they cannot hear each others transmission; see Fig. 1(b).

For the system model presented above we are interested in determining the number of time steps required to gossip in connected configurations. We assume that time is slotted and the system is synchronous. Furthermore, we assume that a transmission lasts for exactly one slot.

2.1. Notation

We will use the following notation frequently in the following discussion. Let $\Omega_n = [0, L_n]^n$ represent the set of all possible placements of n nodes on the line segment $[0, L_n]$ and let $C_n \subset \Omega_n$ represent the set of connected configurations. Let $\tilde{P}_n$ denote the product measure on Ω_n with uniform marginals and define $P_n = \tilde{P}_n(\cdot \mid C_n)$.

3. Results on broadcasting

The gossiping algorithm presented in the next section utilizes a *broadcasting* algorithm in two of its phases. We therefore present a broadcasting algorithm in this section and summarize some of the relevant complexity results. Our detailed analysis of broadcasting in $[0, L]$ has been presented in an earlier paper [7].

Algorithm 1. Let the node that begins the broadcast be represented as x_0 located at position 0 on the line. Node x_0 transmits at time step 1. Identify nodes $x_1, x_2, \ldots, x_k$ with the property that x_i is the most distant node to the right of x_{i-1} still within transmission range of x_{i-1}, i.e., $x_{i-1} < x_i < x_{i-1} + 1$. Therefore, x_k is the *last* node to the right of x_0.

The sequence of transmissions is now $x_0, x_1, \ldots, x_{k-1}$. In time step 1, node x_0 transmits. In time step 2, the node x_1 transmits, and so on. At the end of k time steps (when node x_{k-1} transmits) all the nodes have received the message. Thus the number of time steps required is exactly k.

In [7] we proved that the broadcasting algorithm is optimal and we derived expressions for the expected number of steps required broadcast. In addition, we studied the asymptotic complexity of broadcasting as $n \to \infty$ and $L = f(n)$. These results are summarized in Table 1.

4. Gossiping

Gossiping refers to the problem of efficient *all-to-all* broadcast between a set of nodes where each node is initially assumed to possess a unique secret. The gossiping problem is more complicated than the broadcasting problem because of the possibility of simultaneous transmissions by several nodes. However, as in the case of

Table 1
Asymptotic results for time to broadcast (B_n)

$L \sim n^\alpha$, $0 < \alpha < 1/2$	$\lim_{n\to\infty} E[B_n]/L = 1$	Convergence in expectation
$L \sim n^\alpha$, $0 < \alpha < 1$	$\lim_{n\to\infty} P_n(\lvert B_n/L - 1\rvert > \varepsilon) = 0$	Convergence in probability
$L \sim n\alpha$, $\alpha \leqslant 3/4$	$\lim_{n\to\infty} P_n(\lvert B_n/L - 1\rvert > \varepsilon) = 0$	**Conjecture**
$L \sim n\alpha$, $\alpha > 3/4$	$\lim_{n\to\infty} P_n(\lvert B_n/L - 3/4\alpha\rvert > \varepsilon) = 0$	Convergence in probability

broadcasting we are interested in developing an algorithm to gossip in the minimum number of time steps. As in many papers on gossiping, we assume that during a transmission by a node (which lasts one time step) *all* the secrets it possesses are transmitted.

We present a gossiping algorithm in the next section. In the section on asymptotic results, we derive a *lower bound* for the complexity of gossiping and show that our algorithm achieves this lower bound asymptotically (as $n \to \infty$), thus proving that it is *asymptotically optimal.*

4.1. Algorithms

The algorithm below assumes that time is slotted, i.e., there is a global clock that keeps time and every node has access to it. This assumption is not uncommon in the communications literature, see for instance [12]. We further assume that every transmission lasts for exactly one time slot. Finally, we assume that every node knows the coordinate of every other node that is within transmission distance 1.

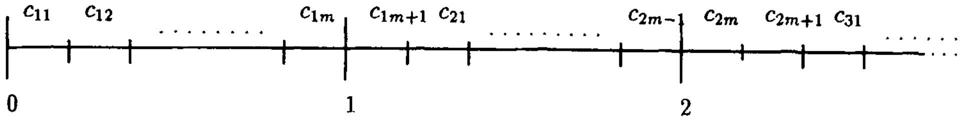
4.1.1. Overview

The gossiping algorithm given below proceeds in three distinct stages. The basic idea is to first allow as many nodes as possible to transmit simultaneously so that all the secrets are made known to a smaller set of nodes (taken together). A *broadcast* is then initiated from both ends of the configuration towards the middle. As these broadcasts proceed, every node transmits all secrets it knows. Thus, eventually, some set of nodes are the first to hear both the broadcasts and *know* all the secrets. This set of nodes does not forward the two broadcasts but instead initiates a new broadcast containing all the secrets and moving simultaneously in both directions, outward.

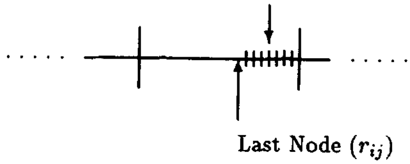
In Section 4.1.2 we present our gossiping algorithm. This algorithm is distributed in the sense that each node only needs to keep track of transmission activity of nodes that it can hear (i.e., that are within distance 1). In Section 4.1.3 we present a local algorithm that every node executes in order to make the gossiping algorithm work.

4.1.2. Gossiping algorithm

Given a *connected* configuration of nodes on $[0, L]$, let us divide it into clusters of size $1 + 1/m$. Each cluster is further divided into smaller subclusters of size $1/m$ (for



(a) Clusters.



(b) Subcluster c_{ij}.

Fig. 2. Structure for gossiping.

some m), see Fig. 2(a). Let us label these subclusters $c_{11}, c_{12}, \ldots, c_{1,m+1}, c_{21}, \ldots$. If x_k denotes the position of the rightmost node in the system and l is an integer such that $(m+1)(l-1)/m < x_k \leqslant (m+1)l/m$ then the set of subclusters is $c_{11}, c_{12}, \ldots, c_{l-1,1}, \ldots, c_{l,m+1}$. Let us assume that r_{ij} denotes the *rightmost* node in a subcluster c_{ij}; see Fig. 2(b).

Algorithm 2.

Stage 1: During this stage all the nodes, in each cluster i get to transmit their secrets. However, in order to avoid the possibility of collisions, we sequence transmissions utilizing the subcluster structure shown in Fig. 2(a). During each step j below, node r_{ij} in c_{ij} hears all transmissions made by nodes in c_{ij} correctly.

Step 1: The rightmost node r_{i1} in subcluster c_{i1} transmits a SIGNAL first. Next all the other nodes in c_{i1} transmit in a left to right order. All these transmissions are *correctly* received by node r_{i1}.This happens in all subclusters c_{i1}, $1 \leqslant i \leqslant l$.

Step 2: The rightmost node r_{i2} transmits a SIGNAL first. This is followed by transmissions from all nodes in c_{i2} in a left to right order. This happens in all subclusters c_{i2}, $1 \leqslant i \leqslant l$.

$\vdots$

Step $m+1$: The rightmost node $r_{i,m+1}$ transmits a SIGNAL first. All remaining nodes in subcluster $c_{i,m+1}$ transmit next in a left to right order. All these transmissions are received with no collisions by node $r_{i,m+1}$.

At the end of Stage 1, all the secrets of each cluster i are concentrated in the nodes r_{ij}, $1 \leqslant j \leqslant m+1$.

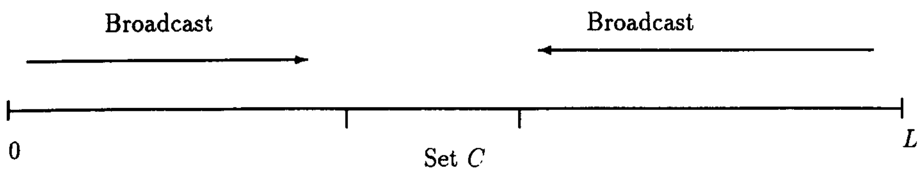


Fig. 3. Stage 3 of the algorithm.

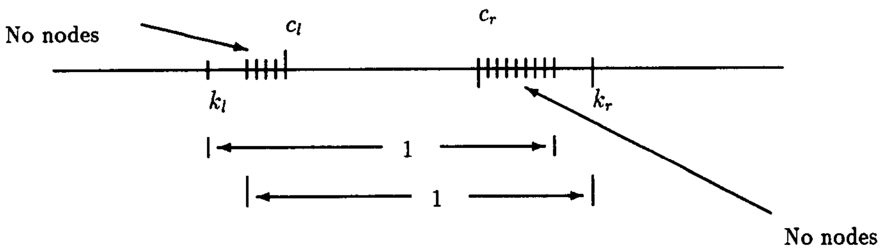


Fig. 4. The rightmost and leftmost nodes in C.

Stage 2: The secrets collected by the nodes r_{ij} in Stage 1 are disemminated to all nodes within distance 1 *to the right* so that the two broadcasts initiated in Stage 3 can "collect" all the secrets as they proceed.

For this stage of the algorithm we first rename the nodes r_{ij}. If r_{ij} lies within the line segment $[k-1, k]$ and is the lth such node in that segment then it is renamed s_{kl}. Therefore, all nodes $r_{1,j}$, $1 \leqslant j \leqslant m$ in $[0, 1]$ are renamed $s_{1,l}$, $1 \leqslant l \leqslant m$. Nodes $r_{1,m+1}, r_{2,1}, r_{2,2}, \ldots, r_{2,m-1}$ that lie within $[1, 2]$ are renamed $s_{2,1}, s_{2,2}, s_{2,3}, \ldots, s_{2,m}$, respectively. After Stage 1 transmissions have completed, all nodes s_{kl} transmit in a left to right order either in *odd* numbered time steps if k is odd or in *even* numbered time steps if k is even. The transmission by node s_{kl} is heard correctly atleast by node $s_{k,l+1}$. Therefore, when node $s_{k,l+1}$ transmits it includes all the secrets contained in s_{kl}s transmission and its own locally held secrets.

This stage takes at most $2m$ time steps and is discussed in more detail in Section 4.1.3 where we discuss the local algorithm executed by each node to determine when to transmit.

Stage 3: After Stage 2 transmissions have ended, a simultaneous broadcast is initiated from *both* ends of the configuration toward the opposite end. As a broadcast proceeds inwards, each node that forwards the broadcast transmits all the information it has (i.e., all the secrets it received in Stage 2 plus the secrets contained in the broadcast it is forwarding), see Fig. 3.

Eventually, both the broadcasts "meet" at some set of nodes C; see Fig. 4. At this point, the two broadcasts are not forwarded anymore. Instead, the secrets contained in both broadcasts are combined with the secrets held by nodes in C and two new broadcasts are initiated proceeding outwards. The purpose of these broadcasts is to *disseminate* all the secrets to all the nodes thus completing gossiping. The nodes that initiate the final pair of broadcasts (to disseminate the secrets) are uniquely determined as follows. All the nodes in C lie within distance 1 of each other (because all of them received *both* the initial broadcasts) and within distance 1 of the nodes that last forwarded the initial pair of broadcasts (call them k_l and k_r). Because each node includes its ID in every transmission, every node in C knows the position of k_l and k_r. Using this information, it is easy for all nodes in C to determine the *leftmost* and *rightmost* nodes, c_l and c_r, respectively, in C; see Fig. 4. Nodes c_l and c_r initiate the last pair of broadcasts to disseminate all secrets.

It is easy to see how the algorithm works. In Stage 1 all nodes in c_{ij} transmit and these transmissions are received correctly by the nodes $r_{i,j}$. During Stage 2 each of these nodes gets to transmit *all the information they have collected* with the guarantee that all nodes within distance 1 to the *right* receive the transmission correctly (i.e., no collisions).

The purpose of Stage 2 of the algorithm is to guarantee that during the broadcast phase of Stage 3 all the information gets collected at the set of nodes C. The broadcast in Stage 3 (see Fig. 3) is initiated at both ends of the configuration and is as presented in Algorithm 1. Since the maximum distance a broadcast can travel in one time step is 1, it "picks up" all the information of all the nodes. This is so because in Stage 2 all information in cluster i gets transmitted to all nodes to the right within distance 1 and it is easy to see that any broadcast in Stage 3 *must* be forwarded by at least one node from this set of nodes. Finally, the last pair of broadcasts in Stage 3 disseminates all the secrets to all nodes.

4.1.3. Local algorithm

In order for the above algorithm to work correctly, it is necessary for each node to keep track of local transmission activity and use this information to transmit at the appropriate time steps. Let us assume that every node is aware of the positions of all nodes that are within distance 1. This information gives node x a view of the system shown in Fig. 5.

Let us now see how nodes use this information to determine when to transmit. Consider Stage 1 of the gossiping algorithm. The idea of each time step here is to allow all nodes in some subcluster c_{ij} to transmit their secrets to r_{ij} without collisions occurring at node r_{ij}. This means that simultaneous transmissions can take place only from nodes that are at distance greater than 1 from node r_{ij}. This property has to hold true for all steps in Stage 1.

Consider the subcluster c_{ij}. Transmissions in c_{ij} begin after r_{ij} (the rightmost node in c_{ij}) transmits. This transmission from r_{ij} is a SIGNAL. The algorithm followed by r_{ij} to determine when to initiate transmissions is the following.

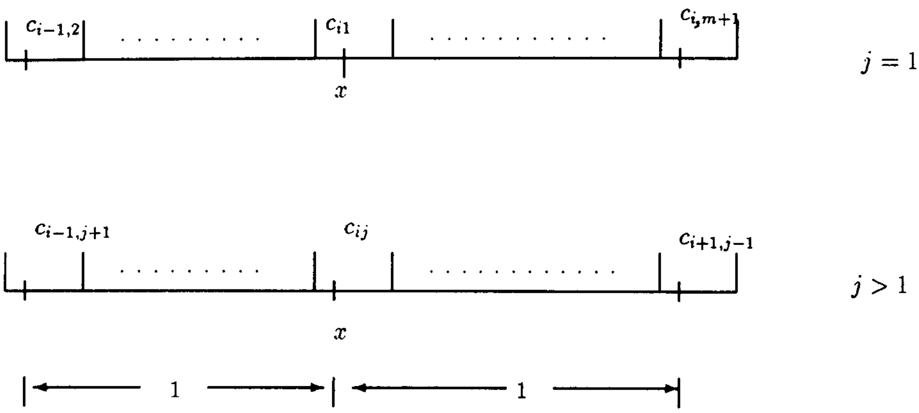


Fig. 5. System view of node x.

Algorithm 3 (for Stage 1).

- r_{ij} knows the number of nodes that lie in subclusters $c_{i+1,k}$, $k \leqslant j-1$ that are within distance 1. Let this number be denoted by a_{ij}. Note that $a_{ij} \leqslant \sum_{k=1}^{j-1} n_{i+1,k}$,
- after gossiping is initiated, r_{ij} counts the number of transmissions from nodes in subclusters $c_{i+1,k}$, $0 < k \leqslant j-1$. This is done by the following method. During each time step, r_{ij} hears one of four things – collision, silence, transmission from a node in $c_{i+1,k}$ or transmission from some node in c_{il}, $0 < l < j$,
  - a *collision* is caused when there are simultaneous transmissions from a node in subcluster $c_{i+1,k}$ and a node in some subcluster c_{il},
  - a *silence* is caused when transmissions in some subcluster $c_{i+1,k}$ are blocked by transmissions in $c_{i+2,k-1}$ (when there are lots of nodes in this subcluster) and this may in turn cause transmissions to be blocked in $c_{i,k+1}$,
  - *successful* transmissions are heard when there is only one transmission during that time step and the origin of the transmission is contained in the transmission itself.
- when a_{ij} = number of collisions + number of successful transmissions from cluster c_{i+1}, node r_{ij} transmits the SIGNAL.

Clearly, after a_{ij} transmissions have occurred in cluster c_{i+1} any further transmissions there will not be heard by r_{ij}. Therefore, all subsequent Stage 1 transmissions to node r_{ij} will be collision free. There is no possibility of collisions occurring at r_{ij} from the left because nodes in the subclusters of cluster c_{i-1} also follow the same algorithm and "lag" behind transmissions in cluster c_i.

Let us now consider Stage 2 of the gossiping algorithm. In order for transmissions to be collision free, we need to sequence transmissions from nodes s_{kl} locally.

Algorithm 4 (for Stage 2).

1. When $s_{k,m}$, the last node in segment $[k-1,k]$ detects that all Stage 1 transmissions (within distance 1) have completed, it transmits a SIGNAL message (to node s_{k1}). Node $s_{k,m}$ transmits its signal in an *odd* numbered time step if k is odd otherwise it does so in an *even* numbered time step.
2. After node s_{k1} detects the signal from $s_{k,m}$, it waits until all Stage 1 transmissions from nodes to its left have completed. Then it transmits all its secrets in the next *odd* or *even* numbered time step, depending on if k is odd or even.
3. The transmission from s_{k1} is received correctly by s_{k2}. Node s_{k2} then transmits in the next *odd* or *even* numbered time step and so on until eventually $s_{k,m-1}$ transmits to $s_{k,m}$.
4. After $s_{k,m}$ detects a Stage 2 transmission from $s_{k+1,1}$ it transmits all its accumulated secrets in the next *odd* or *even* numbered time step. This guarantees that all nodes to its right (and left) within distance 1 receive all the secrets that were originally present in the nodes of segment $[k-1,k]$.

Note that it is easy for node $s_{k,m}$ to detect when all Stage 1 transmissions to its right (within distance 1) have ended by using a method similar to the one described in Algorithm 3. A similar method is used by node s_{k1} to detect when all Stage 1 transmissions to its left have ended.

The SIGNAL from $s_{k,m}$ ensures that all Stage 2 transmissions within $[k-1,k]$ are heard correctly by all nodes within that segment and that they do not clash with Stage 1 transmissions occurring at subclusters to the right within distance 1. Note that without the signal, node s_{k1} will not know when to start Stage 2.

The reason for transmitting in odd or even numbered time steps is to ensure that if Stage 2 transmissions occur simultaneously in neighboring segments then there are no collisions (especially in step 4 of Algorithm 4 above).

Finally, consider Stage 3 of the above algorithm. Stage 3 broadcasts are initiated at the left by node $s_{1,m}$ and from the right by node $s_{L_n,m}$ after they have completed Stage 2 transmissions locally. Observe that these nodes may initiate the broadcasts even while Stage 2 (or even Stage 1) transmissions are going on elsewhere in the system. An intermediate node simply does not forward the Stage 3 broadcast until all Stage 2 transmissions have completed locally. This node can easily determine when Stage 2 transmissions have ended locally by keeping track of local transmissions as in Algorithm 3.

4.2. Complexity

Let n_{ij} denote the number of nodes in the interval c_{ij} for a given configuration $\omega \in C_n$, where C_n represents the set of all connected configurations on $[0,L]$. In order to calculate the complexity of gossiping, let us first compute the number of time steps taken to complete Stage 1.

Complexity of Stage 1: Consider some specific cluster c_i. Let T_i denote the time for all Stage 1 transmissions in c_i to complete. This time is computed by the following inductive procedure:

- The number of time steps, t_{i1}, needed for all Stage 1 transmissions to complete in c_{i1} is

$$t_{i1} = n_{i1}.$$

- The number of time steps taken by $c_{i,j+1}$ to complete all Stage 1 transmissions is

$$\begin{aligned} t_{i,j+1} &= t_{i+1,j} - m_{i+1,j} + n_{i,j+1} && \text{if } t_{i+1,j} - m_{i+1,j} > t_{i,j}, \\ &= n_{i,j+1} + t_{i,j} && \text{otherwise,} \end{aligned}$$

  where, $m_{i+1,j}$ is the number of nodes in $c_{i+1,j}$ that are located at a distance greater than 1 from $r_{i,j+1}$.
- Let l_i denote the last non-empty subcluster in c_i ($l_i \leqslant m + 1$). Then,

$$T_i = t_{i,l_i}.$$

  The complexity of completing Stage 1 is thus,

$$\delta_{m,n} = \max_i \{T_i\}.$$

  A simple upper bound for $\delta_{m,n}$ is obtained as follows:
  Let,

$$M_{m,n} = \max_{i,j} n_{ij}$$

and

$$\Delta_{m,n} = (m + 1) M_{m,n}.$$

Then it is easy to see that

$$\delta_{m,n} \leqslant \Delta_{m,n}.$$

Complexity of Stage 2: After $t_{k,m}$ has signalled, it takes m odd (or even) numbered time steps for transmissions to complete if k is odd (or even). This gives a total complexity of $2m + 2$. Notice that this is a bound on the complexity of Stage 2 because Stages 1 and 2 transmissions can occur simultaneously so long as the transmitting nodes are separated by a distance greater than 2.

Complexity of Stage 3: Observe that in Stage 3 two simultaneous broadcasts are initiated from either end of the configuration. These broadcasts meet at some set of nodes C and continue on to the opposite end of the configuration. The time for this stage to complete is therefore B_n.

Let $D_{m,n}$ represent the overall complexity of gossiping. Then it is easy to see that

$$D_{m,n} = \delta_{m,n} + (2m + 2) + B_n.$$

Using the upper bound for $\delta_{m,n}$ we can bound $D_{m,n}$ as

$$D_{m,n} \leqslant \Delta_{m,n} + (2m + 2) + B_n.$$

5. Asymptotic results

In this section we characterize the asymptotic complexity of gossiping as a function of $L(n)$. We consider the cases when $L(n) \sim n^\alpha$ and $L(n) \sim \alpha n$. We first derive a *lower bound* for gossiping and conclude that our algorithm has a complexity that is asymptotically (i.e., as $n \to \infty$) optimal (in the sense indicated later). All the complexity results are summarized in Table 2.

Theorem 1 (Lower bound for time needed to gossip). *Let G_n be the random variable which denotes the minimum number of time steps required to gossip when there are n nodes in the system. Let,*

$$\beta_n = \frac{1}{\lceil n / \lceil L_n \rceil \rceil}.$$

Then,

(a) $G_n > 1/\beta_n$ *if L is bounded,*

(b) $\lim_{n\to\infty} P_n(G_n > 1/\beta_n + B_n - \log n) = 1$ *if* $\lim_{n\to\infty} L_n/n^\alpha = c_\alpha > 0,\ 0 < \alpha < 1,$

(c) $G_n \geqslant B_n$ *if* $\lim_{n\to\infty} \beta_n = c < \infty,$

where B_n is the time required to broadcast.

Proof. In gossiping, each node has to transmit at least once. When a node transmits, its message is heard by at least one other node (otherwise that transmission is useless). Therefore, we require at least $1/\beta_n$ time steps for every node to transmit once.

Case a: The result follows from the remark above.

Case b: For any sequence of transmissions, observe that at the end of $1/(\beta_n - 1)$th step at least L_n nodes have not transmitted their information. At least one of these L_n nodes lies within a distance $\log n$ to the right of 0 with a probability approaching 1. To see this, consider the segments $[0,1]$, $[1,2], \ldots, [\log n - 1, \log n]$. We now compute the probability of the event M that each of these segments has fewer than n/L_n nodes. Let M_i be the event that the ith segment has fewer than n/L_n nodes. Observe that

$$\lim_{n\to\infty} P_n(M_1) \leqslant \tfrac{1}{2}.$$

The probability mass function of M_2 given M_1 is binomial with mean $(n-k_1)/(L_n-1)$ where $k_1 \leqslant n/L_n$ is the number of nodes in $[0,1]$. It is easy to see that

$$\frac{n-k_1}{L_n-1} \geqslant \frac{n}{L_n}.$$

Therefore,

$$\lim_{n\to\infty} P_n(M_2 \mid M_1) \leqslant \tfrac{1}{2}.$$

Similarly, it can be shown that

$$\lim_{n\to\infty} P_n(M_i \mid M_1, M_2, \ldots, M_{i-1}) \leqslant \tfrac{1}{2}.$$

Therefore,

$$\lim_{n\to\infty} P_n(M) = \lim_{n\to\infty} P_n(M_1)P_n(M_2 \mid M_1)\cdots P_n(M_{\log n} \mid M_1, \ldots, M_{\log n - 1}) = 0.$$

The minimum number of steps required to get this node's secret all the way across the configuration is then $B_n - \log n$. This yields a total number of steps equal to $n/L + B_n - \log n$.

Case c: Trivial because time to gossip is always bounded by time to broadcast. □

Recall that $\Omega_n = [0, L_n]^n$ represents the set of all possible placements of n nodes on the line segment $[0, L_n]$ and $C_n \subset \Omega_n$ represents the set of connected configurations. $\tilde{P}_n$ denotes the product measure on Ω_n with uniform marginals and $P_n = \tilde{P}_n(\cdot \mid C_n)$.

Lemma 1. *Let $L_n/n^\alpha \to c > 0$ as $n \to \infty$, $0 \leqslant \alpha < 1$. Then,*

$$\tilde{P}_n(C_n) \to 1 \quad \text{as } n \to \infty.$$

Proof. Divide the interval $[0, L]$ into intervals of length $\frac{1}{2}$. Let $0 = d_0 < d_1 < d_2 < \cdots < d_{k_n} = L_n$, where $d_1 = \frac{1}{2}$, $d_2 = 1, \ldots, d_k = k/2$ and $k_n = 2L_n$. Let $I_j = (d_{j-1}, d_j]$, $1 \leqslant j \leqslant k_n$. Let $D_k(n) = \{w \in \Omega_n \mid \text{there is at least one node in } I_k\}$ and let D_k^c denote its complement. It is easy to see that,

$$\bigcap_{k=1}^{k_n} D_k(n) \subseteq C_n.$$

We now show that $\tilde{P}_n(\bigcup_1^{k_n} D_k^c(n)) \to 0$ as $n \to \infty$.

$$\tilde{P}_n\left(\bigcup_1^{k_n} D_k^c(n)\right) \leqslant \sum_1^{k_n} \tilde{P}_n(D_k^c(n)),$$

$$\tilde{P}_n(D_k^c(n)) = \left(\frac{L_n - 1/2}{L_n}\right)^n = \left(1 - \frac{1}{2L_n}\right)^n.$$

Since $L_n \sim n^\alpha$ and $k_n = 2L_n$ it easily follows that

$$\sum_1^{k_n} \tilde{P}_n(D_k^c(n)) \to 0 \text{ as } n \to \infty .$$

This proves Lemma 1. $\square$

In the theorem below we prove that the complexity of gossipping approaches $n/L + B_n$ asymptotically when $L_n \sim n^\alpha$.

Theorem 2. *If $L \sim n^\alpha$, $0 \leqslant \alpha < 1$, then $\forall \varepsilon > 0$,*

$$\lim_{m\to\infty} \lim_{n\to\infty} P_n\left(\frac{\Delta_{m,n} + 2m + B_n}{n/L_n + B_n} > 1 + \varepsilon\right) \to 0.$$

Proof. We choose $m > 2/\varepsilon$ in all the three cases that follow. For a given ε, m is a fixed number and therefore it is sufficient to show that

$$\lim_{m\to\infty} \lim_{n\to\infty} P_n\left(\frac{\Delta_{m,n} + B_n}{n/L_n + B_n} > 1 + \varepsilon\right) \to 0.$$

Case 1: $\alpha < \frac{1}{2}$.

$$P_n\left(\frac{\Delta_{m,n} + B_n}{n/L_n + B_n} > 1 + \varepsilon\right) = P_n\left(\frac{\Delta_{m,n} + B_n}{n/L_n(1 + B_n/(n/L_n))} > 1 + \varepsilon\right).$$

Since $B_n/L_n \xrightarrow{P} 1$, we have $B_n/(n/L_n) \xrightarrow{P} 0$ (remember $\alpha < \frac{1}{2}$). It is therefore sufficient to estimate,

$$P_n\left(\frac{\Delta_{m,n}}{n/L_n} > 1 + \varepsilon\right).$$

From Lemma 1 it follows that it is sufficient to estimate,

$$\tilde{P}_n\left(\frac{\Delta_{m,n}}{n/L_n} > 1 + \varepsilon\right).$$

We observe that $E[n_{ij}] = E[n_{kl}]$ for all clusters c_{ij} and c_{kl} (E stands for expectation w.r.t. $\tilde{P}_n$). Since

$$\sum_{i,j} n_{ij} = n,$$

we have

$$E[n_{ij}] = \frac{n}{mL_n}, \quad \forall i,j.$$

Using the fact that the ($\tilde{P}_n$) marginal distribution of n_{ij} is binomial with a probability of success $1/mL_n$ and number of trials n we estimate,

$$\begin{aligned}\tilde{P}_n\left(\Delta_{m,n} > \frac{n}{L_n} + \frac{\varepsilon n}{L_n}\right) &= \sum_{i,j} \tilde{P}_n\left(n_{ij} > \frac{n}{(m+1)L_n} + \frac{\varepsilon n}{(m+1)L_n},\ n_{ij} = M_{m,n}\right)\\ &\leqslant mL_n\tilde{P}_n\left(n_{11} > \frac{n}{(m+1)L_n} + \frac{\varepsilon n}{(m+1)L_n}\right)\\ &\leqslant mL_n\tilde{P}_n\left(n_{11} > \frac{n}{mL_n} + \frac{\varepsilon n/2}{(m+1)L_n}\right)\\ &\leqslant \frac{4(m+1)^2}{\varepsilon^2}\frac{L_n^2}{n} \xrightarrow{n\to\infty} 0.\end{aligned}$$

Case 2: $\alpha > \frac{1}{2}$. We can write,

$$\begin{aligned}&P_n\left(\frac{\Delta_{m,n} + B_n}{n/L_n + B_n} > 1 + \varepsilon\right)\\ &= \sum_{i,j} P_n\left(\frac{n_{ij}(m+1) + B_n}{n/L_n + B_n} > 1 + \varepsilon,\ M_{m,n} = n_{ij}\right)\\ &= \sum_{i,j} P_n\left(\frac{(n_{ij}(m+1) - (1+1/m)n/L_n) + B_n + n/L_n(1+1/m)}{n/L_n + B_n} > 1 + \varepsilon,\right.\\ &\qquad\qquad \left. M_{m,n} = n_{ij}\right)\\ &\leqslant mL_n\tilde{P}_n\left(\frac{(m+1)(n_{11} - n/(mL_n)) + (B_n + n/L_n(1+1/m))}{n/L_n + B_n} > 1 + \varepsilon\right).\end{aligned}$$

The probability above may be written as

$$\tilde{P}_n\left(\frac{(m+1)/L_n(n_{11} - n/(mL_n)) + (B_n/L_n + n/L_n^2(1+1/m))}{n/L_n^2 + B_n/L_n} > 1 + \varepsilon\right).$$

Now using the fact that $B_n/L_n \xrightarrow{P} 1$, $n/L_n^2 \to 0$ it is sufficient to estimate,

$$\tilde{P}_n\left(n_{11} - \frac{n}{mL_n} > \frac{L_n\varepsilon}{(m+1)}\right) \leqslant \frac{n/(mL_n)}{\varepsilon^2 L_n^2}(m+1)^2 = \frac{n}{\varepsilon^2 m L_n^3}(m+1)^2.$$

Therefore,

$$\begin{aligned}&mL_n\tilde{P}_n\left(\frac{(m+1)(n_{11} - n/(mL_n)) + B_n + n/L_n(1+1/m)}{n/L_n + B_n} > 1 + \varepsilon\right)\\ &\leqslant \frac{n}{\varepsilon^2 L_n^2}(m+1)^2 \to 0 \text{ as } n \to \infty.\end{aligned}$$

This proves the theorem for $\alpha > \frac{1}{2}$.

Case 3: $\alpha = \frac{1}{2}$. As above, we need to estimate,

$$
\begin{aligned}
&P_n\left(\frac{\Delta_{m,n} + B_n}{n/L_n + B_n} > 1 + \varepsilon\right)\\
&= P_n\left(\left(\Delta_{m,n} - \frac{(m+1)n}{mL_n}\right) + \left(\frac{(m+1)n}{mL_n} - \frac{n}{L_n}\right) > \varepsilon\left(\frac{n}{L_n} + B_n\right)\right)\\
&\leqslant mL_n\tilde{P}_n\left(\left((m+1)n_{11} - (m+1)\frac{n}{mL_n}\right) > \frac{\varepsilon}{2}\left(\frac{n}{L_n} + B_n\right)\right)\\
&= mL_n\tilde{P}_n\left(\left(n_{11} - \frac{n}{mL_n}\right)^4 > \left(\frac{\varepsilon}{2(m+1)}\right)^4\left(\frac{n}{L_n} + B_n\right)^4\right).
\end{aligned}
$$

Noting that

$$
E\left(\left(n_{11} - \frac{n}{mL_n}\right)^4\right) \leqslant C_1\frac{n}{mL_n} + C_2\frac{n^2}{(mL_n)^2},
$$

where $C_1, C_2 < \infty$, we have

$$
\begin{aligned}
&mL_n\tilde{P}_n\left(\left(n_{11} - \frac{n}{mL_n}\right)^4 > \left(\frac{\varepsilon}{2(m+1)}\right)^4\left(\frac{n}{L_n} + B_n\right)^4\right)\\
&\leqslant \frac{C_1 n/(mL_n) + C_2 n^2/(mL_n)^2}{(\varepsilon/(2(m+1)))^4 (n/L_n + B_n)^4} mL_n,
\end{aligned}
$$

since $L_n \sim n^{1/2}$ and $B_n \sim n^{1/2}$, we have

$$
mL_n\tilde{P}_n\left(\left(n_{11} - \frac{n}{mL_n}\right) > \frac{\varepsilon}{2(m+1)}\left(\frac{n}{L_n} + B_n\right)\right) \to 0 \quad \text{as } n \to \infty.
$$

This proves Theorem 2. □

Theorem 3 (Asymptotic optimality). $\forall \varepsilon > 0$,

$$
\inf_{m \in \mathcal{N}} \lim_{n \to \infty} P_n(D_{m,n} > (1 + \varepsilon)G_n) = 0.
$$

Remark. What this theorem states is that given an $\varepsilon > 0$, $\exists m \in \mathcal{N}$ s.t. we can find a gossiping algorithm $A(m)$ for which the number of time steps requires to gossip is asymptotically (in n the number of nodes) no more than $(1 + \varepsilon)G_n$ in probability (G_n is the lower bound from Theorem 1).

Proof of Theorem 3. Let,

$$
\lim_{n \to \infty} \frac{L_n}{n^\alpha} = c_\alpha, \quad 0 \leqslant \alpha < 1.
$$

The result in this case follows easily from parts (a) and (b) of Theorems 1 and 2 above.
Let,

$$\lim_{n\to\infty} \frac{L_n}{n} = \alpha > 0.$$

In this case we will prove the result for $m = 1$ and the result would follow from Theorem 1 if we can show,

$$\forall\delta > 0, \quad P_n(\Delta_{1,n} > \delta B_n) \xrightarrow{n\to\infty} 0$$

for some $m \in \mathcal{N}$. Since it can be easily shown that $\forall\alpha > 0$,

$$P_n\left(\frac{B_n}{L_n} > 0\right) \xrightarrow{n\to\infty} 1,$$

it would be sufficient to show that

$$\forall\delta > 0, \quad P_n(\Delta_{1,n} > \lfloor \delta L_n \rfloor) \xrightarrow{n\to\infty} 0.$$

Let Q_n be the restriction of the product measure on $[0,n]^n$ to C_n, the set of connected configurations in $[0,n]$. In [8] we showed that under Q_n the internode distances are i.i.d. (independent identically distributed) uniform random variables. Let N_n be the position of the last node of configurations in $[0,n]^n$. P_n which is the restriction of the product measure on $[0,L_n]^n$ to connected configurations in $[0,L_n]^n$ can be represented as the restriction of Q_n to the set $C_{L_n} = \{\omega \in C_n \,|\, N_n(\omega) \leqslant L_n\}$. Since the distribution of N_n under Q_n is that of a sum of n i.i.d. uniform random variables it is clear that

$$\text{if } \lim_{n\to\infty} \frac{L_n}{n} = \alpha \geqslant \tfrac{1}{2}, \quad \text{then } \lim_{n\to\infty} Q_n(C_{L_n}) > 0,$$

while

$$\text{if } \alpha < \tfrac{1}{2}, \quad \text{then } \lim_{n\to\infty} Q_n(C_{L_n}) = 0.$$

Let n_i be the number of nodes in $[i-1, i)$, $1 \leqslant i \leqslant n$, in a configuration from $[0,n]^n$, and let Y_i, $1 \leqslant i \leqslant n$, represent the internode distances. Since $m = 1$,

$$\begin{aligned} P_n(\Delta_{1,n} \geqslant \lfloor \delta L_n \rfloor) &= Q_n(2\{\max_{1\leqslant i\leqslant n} n_i\} \geqslant \lfloor \delta L_n \rfloor \,|\, N_n \leqslant L_n) \\ &\leqslant \sum_{i=1}^{L_n} Q_n\,(n_i \geqslant \lfloor \delta L_n \rfloor/2 \,|\, N_n \leqslant L_n). \end{aligned}$$

We now estimate $Q_n(n_i \geqslant \delta L_n \,|\, N_n \leqslant L_n)$. Let

$$S(i,j) = \sum_{k=1}^{j} Y_k,$$

then

$$Q_n(n_i \geqslant \lfloor \delta L_n \rfloor | N_n \leqslant L_n)$$
$$= \sum_{j=i-1}^{n-\lfloor \delta L_n \rfloor} Q_n(i-2 \leqslant S(1,j) \leqslant i-1,\, S(1,j+1) \geqslant i-1,$$
$$i < S(1,j+\lfloor \delta L_n \rfloor) < i-1 \,|\, N_n \leqslant L_n).$$

Recalling that under Q_n, Y_ks are i.i.d. random variables, we obtain

$$Q_n(n_i \geqslant \lfloor \delta L_n \rfloor | N_n \leqslant L_n) \leqslant \frac{Q_n(S(1,\lfloor \delta L_n \rfloor) < 1)}{Q_n(S(1,n) \leqslant L_n)}.$$

Case 1: $\alpha \geqslant \frac{1}{2}$. In this case $Q_n(S(1,n) \leqslant L_n) \xrightarrow{n\to\infty} c_\alpha > 0$. Since

$$\frac{S(1,\lfloor \delta L_n \rfloor)}{\lfloor \delta L_n \rfloor} \xrightarrow{P} \frac{\lfloor \delta L_n \rfloor}{2},$$

we see that $\lim_{n\to\infty} Q_n(S(1,\lfloor \delta L_n \rfloor) < 1) = 0$.

Since we have to estimate

$$\sum_{i=1}^{L_n} Q_n(n_i \geqslant \lfloor \delta L_n \rfloor | N_n \leqslant L_n),$$

we need a sharper estimate. From the *large deviation property* for i.i.d. random variables, we have

$$\lim_{n\to\infty} \frac{1}{\lfloor \delta L_n \rfloor} \log\left(Q_n\left(\frac{S(1,\lfloor \delta L_n \rfloor)}{\lfloor \delta L_n \rfloor} \leqslant \beta \right)\right) \leqslant \Lambda(\beta),$$

where

$$\Lambda(x) = \sup\left\{ \lambda x - \left(\log\left(\frac{e^\lambda - 1}{\lambda} \right)\right) \middle| \lambda \in \mathscr{R} \right\}$$

see [2]. It can be easily seen that $\Lambda(x) > 0$ if $x \neq \frac{1}{2}$. Therefore,

$$Q_n(S(1,\lfloor \delta L_n \rfloor) \leqslant 1) \leqslant e^{-c\lfloor \delta L_n \rfloor}, \quad \text{where } c > 0.$$

From this it follows that

$$\sum_{i=1}^{L_n} Q_n(n_i \geqslant \lfloor \delta L_n \rfloor | N_n \leqslant L_n) \leqslant L_n \frac{Q_n(S(1,\lfloor \delta L_n \rfloor) \leqslant 1)}{Q_n(S(1,n) \leqslant n)} \xrightarrow{n\to\infty} 0,$$

since $Q_n(S(1,n) \leqslant n)$ is a constant.

Case 2: $0 < \alpha < \frac{1}{2}$. Since $\lim_{n\to\infty} Q_n(S(1,n) \leqslant L_n) = 0$, we need sharper estimates in this case. It can be shown that the rate function $\Lambda(x)$ defined above is a non-negative strictly decreasing function in $(0,\frac{1}{2})$ and that $\lim_{x\to 0}\Lambda(x) = \infty$. From the classical

Table 2
Complexity of gossiping

$L(n)$	Complexity	Comments
$L \sim n^{\alpha}$, $0 \leqslant \alpha < 1$	$n/L + L + c$	c a constant
$L \sim \alpha n$, $0 < \alpha \leqslant 3/4$	$k + L$	**Conjecture**
$L > \alpha n$, $\alpha > 3/4$	$k + 3L/4\alpha$	k a constant

theorem on large deviations due to Cramér, we have

$$\lim_{n\to\infty} \frac{1}{n} \log Q_n\left(\frac{S(1,n)}{n} \leqslant \frac{L_n}{n}\right) \geqslant -\Lambda(\alpha)$$

and

$$\lim_{n\to\infty} \frac{1}{\lfloor \delta L_n \rfloor} \log Q_n\left(\frac{S(1,\lfloor \delta L_n \rfloor)}{\lfloor \delta L_n \rfloor} \leqslant \beta\right) \leqslant -\Lambda(\beta),$$

since

$$\lim_{x\to 0} \Lambda(x) = \infty, \quad \exists \beta \in (0,\alpha),$$

such that

$$\Lambda(x) > \frac{\Lambda(\alpha)}{\alpha\delta} \quad \text{if } 0 \leqslant x \leqslant \beta.$$

From this it follows that, for large enough n,

$$\frac{Q_n(S(1,\lfloor \delta L_n \rfloor) \leqslant 1)}{Q_n(S(1,n) \leqslant L_n)} \leqslant e^{-c\lfloor \delta L_n \rfloor}, \quad \text{where } c > 0$$

from which we obtain

$$\lim_{n\to\infty} \sum_{i=1}^{L_n} Q_n(n_i \geqslant \lfloor \delta L_n \rfloor \mid N_n \leqslant L_n) = 0.$$

This concludes the proof of Theorem 3. □

Thus we conclude that Algorithm 2 is asymptotically optimal for gossiping. The complexity of gossiping is summarized in Table 2.

6. Conclusions and future work

In this paper we introduced a new system model and studied the problem of gossiping. In this model nodes communicate via radio rather than telephone lines. As a result *collisions* become an important consideration in any algorithm. We presented analysis for the number of time steps required to gossip and showed that the complexity to gossip is asymptotically optimal.

Because of the complexity of the new system model, we restricted our attention in this paper to the case when all the nodes are arranged on a line. A generalization of this model is to study systems where the nodes are placed on a two-dimensional plane.

A different kind of generalization is if we assume that the transmission radius is not a constant but a random variable. Transmission signals tend to fade as a function of distance from the transmitter. Perhaps a *normal* distribution for the transmission radius would yield interesting results.

Appendix: Preprocessing stage

The algorithms in Section 4 requires nodes to know the positions of all nodes within distance 1 prior to gossiping. If a *new* node joins an existing system, this information can be passed to it as part of the initialization process. On the other hand, if we assume that initially no node knows the position of any other, then a more complicated approach is needed.

The main problem in exchanging coordinate information is that all the nodes need to be able to transmit their coordinate *without interference* from any other node. Initially, when nodes have no information about any of the others, there is no deterministic way in which transmissions can be sequenced to generate collision-free transmissions. Therefore, we will have to rely on *multiaccess* protocols that work in such environments. One appropriate protocol is called the *splitting algorithm*; see [1].

Let us first consider the case when all nodes lie in the interval $[0, 1]$ only. Initially, all nodes transmit. Following a collision, all nodes toss a coin (with a probability $\frac{1}{2}$ of coming up *heads*) and those that tossed a heads attempt again in the next slot while all those that received a tails on the toss wait for all the nodes that received a heads to finish transmission before attempting to transmit themselves. This algorithm is applied recursively at every stage until all the nodes have transmitted successfully.

Now let us assume that the connected configuration stretches from 0 to L. Then the algorithm to exchange coordinate information proceeds as follows.

Algorithm 5.

1. All nodes in intervals $[0, 1]$, $[2, 3]$, $[4, 5]$, ... use the splitting algorithm to transmit their coordinates to all others in that interval.
2. All nodes in the intervals $[1, 2]$, $[3, 4]$, ... use the splitting algorithm to transmit their coordinates to all others in that interval.
3. (a) Nodes in intervals $[0, 1]$, $[3, 4]$, ... transmit their coordinates in a left to right order (this is possible because all nodes know the positions of every other within its interval).

(b) Nodes in intervals $[1,2]$, $[4,5], \ldots$ transmit their coordinates in a left to right order.

(c) Nodes in intervals $[2,3]$, $[5,6], \ldots$ transmit their coordinates in a left to right order.

Clearly all transmissions in step 3 are collision free.

The complexity of the above algorithm for a *specific* connected configuration may be computed as follows. Let T_k denote the expected number of time steps needed by the splitting algorithm to complete in an interval of length 1 containing k nodes ([5] provides complex upper and lower bounds for T_k). Then steps 1 and 2 of Algorithm 5 take T_{n_1} and T_{n_2} steps, respectively, where,

$$n_1 = \max\{m_1, m_3, m_5, \ldots\},$$

$$n_2 = \max\{m_2, m_4, m_6, \ldots\},$$

where m_i = number of nodes in interval $[i-1, i]$.

Step 3 of the algorithm takes $n_1' + n_2' + n_3'$ time steps to complete, where

$$n_1' = \max\{m_1, m_4, \ldots\},$$

$$n_2' = \max\{m_2, m_5, \ldots\},$$

$$n_3' = \max\{m_3, m_6, \ldots\}$$

yielding a total complexity of $T_{n_1} + T_{n_2} + n_1' + n_2' + n_3'$.

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